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Extended Itô calculus for symmetric Markov processes

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Chen, Fitzsimmons, Kuwae and Zhang (*Ann. Probab.* **36** (2008) 931–970) have established an Itô formula consisting in the development of $F(u(X))$ for a symmetric Markov process X , a function u in the Dirichlet space of X and any C^2 -function F . We give here an extension of this formula for u locally in the Dirichlet space of X and F admitting a locally bounded Radon–Nikodym derivative. This formula has some analogies with various extended Itô formulas for semi-martingales using the local time stochastic calculus. But here the part of the local time is played by a process $(\Gamma_t^a, a \in \mathbb{R}, t \geq 0)$ defined thanks to Nakao’s operator (*Z. Wahrsch. Verw. Gebiete* **68** (1985) 557–578).

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1. Introduction and main results

For any real-valued semimartingale $Y = (Y_0 + M_t + N_t)_{t \geq 0}$ (M martingale and N bounded variation process) and any function F in $C^2(\mathbb{R})$, the classical Itô formula

$$\begin{aligned} F(Y_t) = F(Y_0) &+ \int_0^t F'(Y_s) dM_s + \int_0^t F'(Y_s) dN_s + \frac{1}{2} \int_0^t F''(Y_s) d\langle M^c \rangle_s \\ &+ \sum_{s \leq t} \{F(Y_s) - F(Y_{s-}) - F'(Y_{s-}) \Delta Y_s\} \end{aligned} \quad (1.1)$$

provides both an explicit expansion of $(F(Y_t))_{t \geq 0}$ and its stochastic structure of semi-martingale.

Let now E be a locally compact separable metric space, m a positive Radon measure on E , and X a m -symmetric Hunt process. Under the assumption that the associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ of X is regular, Fukushima has showed that for any function u in \mathcal{F} , the additive functional (abbreviated as AF) $(u(X_t) - u(X_0))_{t \geq 0}$ admits the following

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unique decomposition:

$$u(X_t) = u(X_0) + M_t^u + N_t^u \quad \mathbb{P}_x\text{-a.e. for quasi-every } x \text{ in } E, \quad (1.2)$$

where M^u is a martingale AF of finite energy and N^u is a continuous AF of zero energy.

Although $u(X)$ is not in general a semimartingale, Nakao [14] and Chen *et al.* [3] have proved that (1.1) is still valid with $u(X)$, M^u and N^u replacing, respectively, Y , M and N . This is done thanks to the construction of a new stochastic integral with respect to N^u , which takes the place of the well-defined Lebesgue–Stieltjes integral for the bounded variation processes. As the classical Itô formula (1.1), this Itô formula for symmetric Markov processes requires the use of \mathcal{C}^2 -functions.

For the semimartingale case, there exist extended versions of (1.1) relaxing this regularity condition. These extensions are based on the replacement of the fourth and fifth terms of the right-hand side of (1.1) by an alternative expression requiring only the existence of F' and some integrability condition on F' (see, e.g., [7–9]). The integrability condition insures also the existence of the other terms of (1.1).

The question of relaxing the regularity condition on F in the formula of Nakao and Chen *et al.* is a more complex question. Indeed the integral $\int_0^t F'(u(X_s)) dN_s^u$ is well-defined only when $F'(u)$ belongs to \mathcal{F}_{loc} , the set of functions locally in \mathcal{F} . As in [3], $u \in \mathcal{F}_{\text{loc}}$ means that there exists a nest of finely open Borel sets $\{G_k\}_{k \in \mathbb{N}}$ and a sequence $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e. on G_k . As an example, in the case X is a Brownian motion, this condition implies that the second derivative F'' exists at least as a weak derivative. Nevertheless, in the general case, we know that for any function F element of $\mathcal{C}^1(\mathbb{R})$ with bounded derivative, $F(u)$ belongs to \mathcal{F} and the process $F(u(X))$ hence admits a Fukushima decomposition. We can thus hope to obtain an Itô formula for \mathcal{C}^1 -functions F that would express each element of the decomposition of $F(u(X))$ in terms of F , u , N^u and M^u . Our purpose here is to establish such a formula. The obtained formula is actually established for the functions F with locally bounded Radon–Nikodym derivative and u element of \mathcal{F}_{loc} .

Before introducing this extended Itô formula for symmetric Markov processes, remark that one can easily obtain an extended Itô formula in case $u(X)$ is a semimartingale. Indeed, under the assumption that X has an infinite life time, we note (see (3.4) in [3]) that $u(X)$ is then a reversible semimartingale and that one can hence make use of [7] or [10] to develop $F(u(X))$. But in general, $u(X)$ is not a semimartingale.

The extended Itô formula for symmetric Markov processes presented here is based on the construction for a fixed $t > 0$, of a stochastic integral of deterministic functions with respect to the process $(\Gamma_t^a(u))_{a \in \mathbb{R}}$, defined as follows.

For u in \mathcal{F} , let $M^{u,c}$ be the continuous part of M^u . For any real a and $t \geq 0$, we set

$$Z_t^a(u) = \int_0^t 1_{\{u(X_s) \leq a\}} dM_s^{u,c}$$

and define Γ^a by

$$\Gamma^a(u) = (\Gamma_t^a(u))_{t \geq 0} = (\Gamma(Z^a(u))_t)_{t \geq 0} = \Gamma(Z^a(u)),$$

where Γ is the operator on the space of martingale AF with finite energy constructed by Nakao [14] (its definition is recalled in Section 2). The process $(\Gamma_t^a(u))_{t \geq 0}$ is hence an additive functional with zero energy.

In Section 2, we will see that the definition of $\Gamma^a(u)$ can be extended to functions u in \mathcal{F}_{loc} . In that case, the process $M^{u,c}$ is a continuous martingale AF on $\llbracket 0, \zeta \rrbracket$ locally of finite energy and the process $(\Gamma_t^a(u))_{t \geq 0}$ is an AF on $\llbracket 0, \zeta \rrbracket$ locally with zero energy.

As shown by the Tanaka formula (1.4) below, the doubly-indexed process $(\Gamma_t^a(u), a \in \mathbb{R}, t \geq 0)$ plays almost the part of a local time process for $u(X)$. In Section 5, this analogy with local time will be fully clarified under some stronger assumption on u .

To introduce the obtained Itô formula, we need the objects presented by the following lemma. We denote by $(N(x, dy), H)$ a Lévy system for X (See Definition A.3.7 of [12]), by ν_H the Revuz's measure of H and by ζ the life time of X .

Lemma 1.1. *Let $u \in \mathcal{F}$ (resp., $u \in \mathcal{F}_{\text{loc}}$). There exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to 0 and such that for any locally absolutely continuous function F from \mathbb{R} into \mathbb{R} with a locally bounded Radon–Nikodym derivative, the following two processes are well-defined.*

$$\begin{aligned} M_t^d(F, u) &= \lim_{n \rightarrow \infty} \left\{ \sum_{s \leq t} \{F(u(X_s)) - F(u(X_{s-}))\} 1_{\{\varepsilon_n < |u(X_s) - u(X_{s-})| < 1\}} 1_{\{s < \zeta\}} \right. \\ &\quad \left. - \int_0^t \int_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \{F(u(y)) - F(u(X_s))\} N(X_s, dy) dH_s \right\} \\ A_t(F, u) &= \lim_{n \rightarrow \infty} \int_0^t \int_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \{F(u(y)) - F(u(X_s))\} N(X_s, dy) dH_s. \end{aligned}$$

The above limits are uniform on any compact of $[0, \infty)$ (resp., $[0, \zeta)$) \mathbb{P}_x -a.e. for q.e. $x \in E$. Moreover, $(M_t^d(F, u))_{t \geq 0}$ is a local martingale AF (resp., AF on $\llbracket 0, \zeta \rrbracket$) with locally finite energy and $(A_t(F, u))_{t \geq 0}$ is a continuous AF (resp., AF on $\llbracket 0, \zeta \rrbracket$) locally with 0 energy.

With the notation of Lemma 1.1, we have the following Itô formula.

Theorem 1.2. *Let $u \in \mathcal{F}$ (resp., $u \in \mathcal{F}_{\text{loc}}$). For any locally absolutely continuous function F from \mathbb{R} into \mathbb{R} with a locally bounded Radon–Nikodym derivative F' such that $F(0) = 0$, the process $(F(u(X_t)), t \in [0, \infty))$ (resp., $t \in [0, \zeta)$) admits the following decomposition \mathbb{P}_x -a.e. for q.e. $x \in E$*

$$F(u(X_t)) = F(u(X_0)) + M_t(F, u) + Q_t(F, u) + V_t(F, u), \quad (1.3)$$

where $M(F, u)$ is a local martingale AF (resp., AF on $\llbracket 0, \zeta \rrbracket$) locally of finite energy, $Q(F, u)$ is an AF (resp., AF on $\llbracket 0, \zeta \rrbracket$) locally of zero energy, and $V(F, u)$ is a bounded

variation process, respectively, given by:

$$\begin{aligned} M_t(F, u) &= M_t^d(F, u) + \int_0^t F'(u(X_s)) dM_s^{u,c}, \\ Q_t(F, u) &= \int_{\mathbb{R}} F'(z) d_z \Gamma_t^z(u) + A_t(F, u), \\ V_t(F, u) &= \sum_{s \leq t} \{F(u(X_s)) - F(u(X_{s-}))\} 1_{\{|u(X_s) - u(X_{s-})| \geq 1\}} 1_{\{s < \xi\}} \\ &\quad - F(u(X_{\xi-})) 1_{\{t \geq \xi\}}. \end{aligned}$$

Note that for u element of \mathcal{F} and F in $\mathcal{C}^2(\mathbb{R})$, (1.3) provides the Itô formula of Chen *et al.* [3] together with the identity connecting integration with respect to $(N_t^u)_{t \geq 0}$ and integration with respect to $(\Gamma_t^a(u))_{a \in \mathbb{R}}$ for smooth enough functions.

As a consequence of Theorem 1.2, we obtain the following Tanaka formula for Γ_t^a :

$$\begin{aligned} \Gamma_t^a(u) &= (u(X_0) - a)^- - (u(X_t) - a)^- + \int_0^t 1_{\{u(X_{s-}) \leq a\}} dM_s^{u,c} \\ &\quad + \lim_{n \rightarrow \infty} \sum_{s \leq t} \{(u(X_s) - a)^- - (u(X_{s-}) - a)^-\} 1_{\{|u(X_s) - u(X_{s-})| > \varepsilon_n\}}, \end{aligned} \tag{1.4}$$

where $(\varepsilon_n)_{n \in \mathbb{N}}$ is the sequence of Lemma 1.1 and the limit is uniform on any compact \mathbb{P}_x -a.e. for q.e. $x \in E$. Using Tanaka's formula for semi-martingales (see [15]), we obtain that when $u(X)$ is a martingale, $-2\Gamma^a(u)$ is the local time process of $u(X)$ at level a . This is the case when $u(x) = x$ and X is a symmetric Lévy process.

Formula (1.3) is hence reminiscent of various extensions of Itô formula involving stochastic integrals with respect to local time, as for example the extensions given in [2] for some martingales, [5] for the Brownian Motion, [6] and [9] for Lévy processes with Brownian component and [16] for Lévy processes without Brownian component. Note that in case the martingale part of $u(X)$ has no continuous component, the process $\Gamma^a(u)$ is identically equal to 0. But (1.3) still represents an improvement of Fukushima's decomposition since (1.3) requires only u in \mathcal{F}_{loc} and F with a locally bounded Radon-Nikodym derivative.

Integration with respect to $(\Gamma_t^a(u))_{a \in \mathbb{R}}$ is constructed in Section 3 and the Itô formula (1.3) is established in Section 4.

In Section 5, we will show that, when $\Gamma(M^{u,c})$ is of bounded variation, $u(X)$ admits a local time process $(L_t^a, a \in \mathbb{R}, t < \zeta)$ satisfying an occupation time formula of the same type as the occupation time formula for the semimartingales and in this case, the process of locally zero energy $Q(F, u)$ can be rewritten as:

$$Q_t(F, u) = -\frac{1}{2} \int_{\mathbb{R}} F'(z) d_z L_t^z + \int_0^t F'(u(X_s)) d\Gamma(M^{u,c})_s + A_t(F, u), \quad t < \zeta.$$

Finally in Section 6 we give a multidimensional version of Theorem 1.2.

2. Preliminaries on m -symmetric Hunt processes

Let E be a locally compact separable metric space, m a positive Radon measure on E such that $\text{Supp}[m] = E$, Δ be a point outside E and $E_\Delta = E \cup \Delta$. Let $X = \{\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \theta_t, \zeta, \mathbb{P}_x, x \in E_\Delta, t \geq 0\}$ be a m -symmetric Hunt Processes such that its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(E; m)$. We may take as Ω the space $D([0, \infty[\rightarrow E_\Delta)$ of càdlàg functions from $[0, \infty[$ to E_Δ , for which Δ is a cemetery (i.e., if $\omega(t) = \Delta$, then $\omega(s) = \Delta$ for any $s > t$) and denote by θ_t the operator $\omega(s) \rightarrow \theta_t \omega(s) := \omega(t + s)$. Every element u of \mathcal{F} admits a quasi-continuous m -version. In the sequel, the functions in \mathcal{F} are always represented by their quasi-continuous m -versions. We use the term “quasi everywhere” or “q.e.” to mean “except on an exceptional set.”

We say that a subset Ξ of Ω is a defining set of a process $A = (A_t)_{t \geq 0}$ with values in $[-\infty, \infty]$, if for any $\omega \in \Xi, t, s \geq 0$: $\theta_t \Xi \subset \Xi$, $A_0(\omega) = 0$, $A_\cdot(\omega)$ is càdlàg and finite on $[0, \zeta[$,

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$$

and $A_t(\omega_\Delta) = 0$, where ω_Δ is the constant path equal to Δ . A (\mathcal{F}_t) -adapted process is an additive functional if it has a defining set $\Xi \in \mathcal{F}_\infty$ admitting an exceptional set, that is, $\mathbb{P}_x(\Xi) = 1$ for q.e. $x \in E$.

An (\mathcal{F}_t) -adapted process is an additive functional on $\llbracket 0, \zeta \rrbracket$ or a local additive functional if it satisfies all the conditions to be an additive functional except that the additive property $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t(\omega))$ is required only for $t + s < \zeta(\omega)$.

Let \mathcal{F}_∞^m (resp., \mathcal{F}_t^m) be the \mathbb{P}_m -completion of $\sigma\{X_s, 0 \leq s < \infty\}$ (resp., $\sigma\{X_s, 0 \leq s \leq t\}$). An (\mathcal{F}_t) -adapted process is an additive functional admitting m -null set if it has a defining set $\Xi \in \mathcal{F}_\infty^m$ such that $\mathbb{P}_x(\Xi) = 1$ for m -a.e. $x \in E$.

The abbreviations AF, PAF, CAF, PCAF and MAF stand respectively for “additive functional,” “positive additive functional,” “continuous additive functional,” “positive continuous additive functional” and “martingale additive functional,” respectively. Let $\mathring{\mathcal{M}}$ and \mathcal{N}_c denote, respectively, the space of MAF’s of finite energy and the space of continuous additive functionals of zero energy N such that $\mathbb{E}_x(|N_t|) < \infty$ q.e. for each $t > 0$. Moreover, $\mathring{\mathcal{M}}^c$ denotes the subset of continuous elements of $\mathring{\mathcal{M}}$ and $\mathring{\mathcal{M}}^d$ denotes the subset of purely discontinuous elements of $\mathring{\mathcal{M}}$.

For $u \in \mathcal{F}$, the elements M^u and N^u of the Fukushima’s decomposition (1.2) are elements of, respectively, $\mathring{\mathcal{M}}$ and \mathcal{N}_c . We denote by $M^{u,c}$, $M^{u,j}$ and $M^{u,\kappa}$, respectively, the continuous part, the jump part and the killing part of M^u (see Section 5.3 of [12]). This three martingales are elements of $\mathring{\mathcal{M}}$.

Let Γ the linear operator from $\mathring{\mathcal{M}}$ to \mathcal{N}_c constructed by Nakao [14] in the following way. It is shown in [14] that for every $Z \in \mathring{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}(w, v) + (w, v)_m = \frac{1}{2} \mu_{\langle M^v + M^{v,\kappa}, Z \rangle}(E) \quad \text{for every } v \in \mathcal{F},$$

where $(w, v)_m = \int_E w(x)v(x)m(dx)$ and $\mu_{\langle M^v + M^{v,\kappa}, Z \rangle}$ is the smooth signed measure corresponding to $\langle M^v + M^{v,\kappa}, Z \rangle$ by the Revuz correspondence. The process $\Gamma(Z)$ is then

defined by:

$$\Gamma_t(Z) = N_t^w - \int_0^t w(X_s) ds.$$

This operator satisfies: $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Thus N^u admits the decomposition:

$$N^u = {}^cN^u + {}^jN^u + {}^\kappa N^u, \quad (2.1)$$

where for $p \in \{c, j, \kappa\}$: ${}^pN^u = \Gamma(M^{u,p})$.

For a Borel subset B of $E \cup \{\Delta\}$, it is known that $\tau_B = \inf\{t > 0: X_t \notin B\}$ and $\sigma_B = \inf\{t > 0: X_t \in B\}$ are (\mathcal{F}_t) -stopping times.

An increasing sequence of Borel sets $\{G_k\}$ in E is called a nest if

$$\mathbb{P}_x\left(\lim_{k \rightarrow \infty} \tau_{G_k} = \zeta\right) = 1 \quad \text{for q.e. } x \in E.$$

Let \mathcal{D} be a class of AF's. We say that an AF (resp., AF on $\llbracket 0, \zeta \rrbracket$) is locally in \mathcal{D} and write $A \in \mathcal{D}_{\text{loc}}$ (resp., $A \in \mathcal{D}_{f\text{-loc}}$) if there exists a sequence $\{A^n\}$ in \mathcal{D} and an increasing sequence of stopping times T_n with $T_n \rightarrow \infty$ (resp., a nest $\{G_n\}$ of finely open Borel sets) such that \mathbb{P}_x -a.e. for q.e. $x \in E$, $A_t = A_t^n$ for $t < T_n$ (resp., $t < \tau_{G_n}$).

Let $\{A^n\}$ be a sequence in \mathcal{D} such that for $k > n$, \mathbb{P}_x -a.e. for q.e. $x \in E$, $A_t^k = A_t^n$ for $t < \tau_{G_n}$, then it is clear that the process

$$A_t := \begin{cases} A_t^n & \text{for } t < \tau_{G_n}, \\ 0 & \text{for } t \geq \zeta \end{cases}$$

is a well-defined element of $\mathcal{D}_{f\text{-loc}}$. A Borel function f from E into \mathbb{R} is said to be locally in \mathcal{F} (and denoted as $f \in \mathcal{F}_{\text{loc}}$), if there is a nest of finely open Borel sets $\{G_k\}$ and a sequence $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$ such that $f = f_k$ q.e. on G_k . This is equivalent to (see Lemma 3.1(ii) in [3]) there is a nest of closed sets $\{D_k\}$ and a sequence $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{F}_b$ such that $f = f_k$ q.e. on D_k . For a such f ,

$$M_t^{f,c} := \begin{cases} M_t^{f_k,c} & \text{for } t < \sigma_{E \setminus G_k}, \\ 0 & \text{for } t \geq \lim_{k \rightarrow \infty} \sigma_{E \setminus G_k} \end{cases}$$

is well defined and belongs to $\mathring{\mathcal{M}}_{f\text{-loc}}$ because, for $n > k$, $M_t^{f_n,c} = M_t^{f_k,c} \forall t \leq \sigma_{E \setminus G_k}$ \mathbb{P}_x -a.e. for q.e. $x \in E$. Indeed, the last property is shown in Lemma 5.3.1 in [12] for τ_{G_k} instead of $\sigma_{E \setminus G_k}$, we conclude with the following observation:

For a CAF A , and a Borel set $G \subset E$, \mathbb{P}_x -a.e. for q.e. $x \in E$:

$$A_t = 0 \quad \text{for } t < \tau_G \quad \Leftrightarrow \quad A_t = 0 \quad \text{for } t < \sigma_{E \setminus G}. \quad (2.2)$$

Every $f \in \mathcal{F}_{\text{loc}}$ admits a quasi-continuous m -version, so we may assume that all $f \in \mathcal{F}_{\text{loc}}$ are quasi-continuous and we set $f(\Delta) = 0$.

We use the following notation for a locally bounded measurable function f and a $(\mathcal{F}_t)_{t \geq 0}$ -semimartingale M :

$$(f * M)_t = \int_0^t f(X_{s-}) dM_s.$$

We will use repeatedly the following fact (see Theorem 5.6.2 in [12]):

For any F in $\mathcal{C}^1(\mathbb{R}^d)$ (d is a positive integer) and u_1, \dots, u_d in \mathcal{F}_b , the composite function $Fu = F(u_1, \dots, u_d)$ belongs to \mathcal{F}_{loc} and

$$M^{Fu, c} = \sum_{i=1}^d F_{x_i}(u) * M^{u_i, c}. \quad (2.3)$$

Chen *et al.* [3] have extended Nakao's definition of the operator Γ to the set of locally square-integrable MAF. We keep using the letter Γ for this extension without possible confusion since thanks to Theorem 3.6 of [3] on the set $\mathring{\mathcal{M}}$, both definitions given in [3] and [14] agree \mathbb{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket$. For a continuous locally square-integrable MAF M , $\Gamma(M)$ is defined to be the following CAF admitting m -null set on $\llbracket 0, \zeta \rrbracket$:

$$\Gamma_t(M) = -\frac{1}{2}(M_t + M_t \circ r_t) \quad \text{for } t \in [0, \zeta[, \quad (2.4)$$

where the operator r_t is defined by

$$r_t(\omega)(s) = \omega((t-s)-)1_{\{0 \leq s \leq t\}} + \omega(0)1_{\{s > t\}} \quad \text{for a path } \omega \in \{t < \zeta\}$$

and $r_t(\omega) := \omega_\Delta$ for a path $\omega \in \{t \geq \zeta\}$.

The continuity of $\Gamma(M)$ \mathbb{P}_m -a.e. on $[0, \zeta[$ is a consequence of Theorem 2.18 in [3].

For f a bounded element of \mathcal{F} and M in $\mathring{\mathcal{M}}$, Nakao has defined the stochastic integral of $f(X)$ with respect to $\Gamma(M)$. We use here the extension of this definition set by Chen *et al.* [3] for f in \mathcal{F}_{loc} and M continuous locally square-integrable MAF as follows:

$$f * \Gamma(M)_t = \int_0^t f(X_{s-}) d\Gamma_s(M) := \Gamma_t(f * M) - \frac{1}{2}\langle M^{f, c}, M \rangle_t. \quad (2.5)$$

It is a CAF admitting m -null set on $\llbracket 0, \zeta \rrbracket$.

When $M \in \mathring{\mathcal{M}}$ and $f \in \mathcal{F}_{\text{loc}}$ the integral $f * \Gamma(M)_t$ can be well defined \mathbb{P}_x -a.e. for q.e. $x \in E$. In particular, the process $(f * \Gamma(M)_t)_{t \geq 0}$ is a local CAF of X (Lemma 4.6 of [3]).

The argument developed by Chen *et al.* to write “q.e. $x \in E$ ” instead of “ m -a.e. $x \in E$ ” in the proof of their Lemma 4.6 in [3], is sufficient to establish Lemma 2.1 below.

Lemma 2.1. *Let A be an AF of X (resp., AF on $\llbracket 0, \zeta \rrbracket$). Let G be a measurable subset of E_Δ (resp., $G \subset E$) and $\Xi := \{\omega \in \Omega: A_t \geq 0, \forall t < \tau_G\}$, then $\mathbb{P}_x(\Xi) = 1$ for m -a.e. $x \in E$ if and only if $\mathbb{P}_x(\Xi) = 1$ for q.e. $x \in E$.*

Lemma 2.2. *Let $\{D_n\}$ be a nest of closed sets and $\sigma := \lim_{n \rightarrow \infty} \sigma_{E \setminus D_n}$. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of $\mathring{\mathcal{M}}^c$ such that for $n < k$, \mathbb{P}_x -a.e. for q.e. $x \in E$, $M_t^n = M_t^k$ if $t < \sigma_{E \setminus D_n}$. Define a continuous locally square-integrable MAF M by:*

$$M_t = \begin{cases} M_t^n & \text{on } t < \sigma_{E \setminus D_n}, \\ 0 & \text{on } t \geq \sigma. \end{cases}$$

Then $\Gamma_t(M)$ can be well defined for all t in $[0, \infty)$ \mathbb{P}_x -a.e. for q.e. $x \in E$, by setting

$$\Gamma_t(M) = \begin{cases} \Gamma_t(M^n) & \text{on } t < \sigma_{E \setminus D_n}, \\ 0 & \text{on } t \geq \sigma. \end{cases} \quad (2.6)$$

Moreover, $\Gamma(M)$ belongs to $\mathcal{N}_{c,f\text{-loc}}$.

For f element of \mathcal{F}_{loc} , (2.5) shows then that $f * \Gamma(M)$ is a well defined CAF on $\llbracket 0, \zeta \rrbracket$.

Proof of Lemma 2.2. A consequence of the m -symmetry assumption on X is that the measure \mathbb{P}_m , when restricted to $\{t < \zeta\}$ is invariant under r_t , so we have \mathbb{P}_m -a.e. on $t < \zeta$: $M_t \circ r_t = M_t^n \circ r_t$ if $t \leq \tau_{D_n} \circ r_t$, but since D_n is closed, for any $\omega \in \Omega$ and $t < \zeta(\omega)$: $t \leq \tau_{D_n}(\omega) \Leftrightarrow t \leq \tau_{D_n}(r_t \omega)$. Hence, it follows from (2.4) that (2.6) hold \mathbb{P}_m -a.e. on $\llbracket 0, \tau_{D_n} \rrbracket$. This shows also, with Lemma 2.1 that if $l > n$, \mathbb{P}_x -a.e. for q.e. $x \in E$: $\Gamma_t(M^n) = \Gamma_t(M^l)$ for $t \leq \tau_{D_n}$ (and consequently for $t \leq \sigma_{E \setminus D_n}$ by (2.2)). Hence, the right-hand side of (2.6) is well defined as a CAF belongs to $\mathcal{N}_{c,f\text{-loc}}$. \square

Remark 2.3. Lemma 2.2 shows that for any $u \in \mathcal{F}_{\text{loc}}$, ${}^c N^u := \Gamma(M^{u,c})$ is an element of $\mathcal{N}_{c,f\text{-loc}}$.

The above Lemma 2.1 and Theorem 4.1 of [3] lead to the following lemma.

Lemma 2.4. *Let M be an element of $\mathring{\mathcal{M}}$ such that $\Gamma(M)$ is of bounded variation on each compact interval of $[0, \zeta[$. Then for every element f of \mathcal{F}_{loc} , \mathbb{P}_x -a.e. q.e. for $x \in E$, on $t < \zeta$, $\int_0^t f(X_s) d\Gamma_s(M)$ coincides with the Lebesgue–Stieljes integral of $f(X)$ with respect to $\Gamma(M)$.*

For the reader's convenience, we recall the following result which is Theorem 5.2.1 of [12] and Theorem 3.2 of [14], the last statement can be seen directly from their proofs. By $e(M)$, we denote the energy of M .

Theorem 2.5. *Let $\{M^n: n \in \mathbb{N}\}$ be a e -Cauchy sequence of $\mathring{\mathcal{M}}$. There exists a unique element M of $\mathring{\mathcal{M}}$ such that $e(M^n - M)$ converges to zero. The subsequence n_k such that there exists $C \in \mathbb{R}_+$ such that for every k in \mathbb{N} : $e(M - M^{n_k}) < C2^{-4k}$, satisfies: \mathbb{P}_x -a.e. for q.e. $x \in E$, $M_t^{n_k}$ and $\Gamma_t(M^{n_k})$ converge uniformly on any finite interval of t to M_t and $\Gamma_t(M)$, respectively.*

3. Integration with respect to Γ^z

We fix a function u of \mathcal{F}_{loc} . Let $\{D_k\}_{k \in \mathbb{N}}$ be a nest of closed sets and $(u_k)_{k \in \mathbb{N}}$ be a sequence of bounded elements of \mathcal{F} associated to u such that $u = u_k$ q.e. on D_k . Let $\sigma := \lim_{n \rightarrow \infty} \sigma_{E \setminus D_n}$. For any real number a , define $Z^a = Z^a(u)$ by

$$Z_t^a = \begin{cases} \int_0^t 1_{\{u_k(X_{s-}) \leq a\}} dM_s^{u_k, c} & \text{for } t \leq \sigma_{E \setminus D_k}, \\ 0 & \text{for } t \geq \sigma. \end{cases}$$

Z^a is a MAF on $\llbracket 0, \zeta \rrbracket$ locally of finite energy. In particular, when u belongs to \mathcal{F} , Z^a is in \mathcal{M}^c for any real a . By Lemma 2.2, $\Gamma(Z^a)$ is well-defined and belongs to $\mathcal{N}_{c, f\text{-loc}}$.

Remark 3.1. For u element of \mathcal{F} , we can choose D_k such that

$$\sigma = \lim_{k \rightarrow \infty} \sigma_{E \setminus D_k} = \infty, \quad \mathbb{P}_x\text{-a.e. for q.e. } x \in E. \quad (3.1)$$

Indeed, in this case, take $u_k := (-k) \vee u \wedge k$ and $G_k := \{x: |u(x)| < k\}$, then it follows from the strict continuity of u that $\lim_{k \rightarrow \infty} \sigma_{E \setminus G_k} = \infty$ \mathbb{P}_x -a.e. for q.e. $x \in E$. Therefore, the nest of closed sets $\{F_k\}_{k \in \mathbb{N}}$ built in the proof of Lemma 3.1(ii) in [3] satisfies the property (3.1) and $u = u_k$ q.e. on F_k . Choose then, $\{D_k\} = \{F_k\}$.

Definition 3.2. The process $(\Gamma_t^a, a \in \mathbb{R}, t \geq 0)$ is defined by $\Gamma_t^a = \Gamma_t^a(u) = \Gamma_t(Z^a)$.

Consider an elementary function f , that is, there exists two finite sequences $(z_i)_{0 \leq i \leq n}$ and $(f_i)_{0 \leq i \leq n-1}$ of real numbers such that:

$$f(z) = \sum_{i=0}^{n-1} f_i 1_{(z_i, z_{i+1}]}(z).$$

For such a function integration with respect to $\Gamma_t = \{\Gamma_t^z; z \in \mathbb{R}\}$ is defined to be the following CAF on $\llbracket 0, \zeta \rrbracket$:

$$\int_{\mathbb{R}} f(z) d_z \Gamma_t^z = \sum_{i=0}^{n-1} f_i (\Gamma_t^{z_{i+1}} - \Gamma_t^{z_i}). \quad (3.2)$$

Thanks to the linearity property of the operator Γ we have for any elementary function f :

$$\int_{\mathbb{R}} f(z) d_z \Gamma_t^z = \Gamma_t \left(\int_0^\cdot f(u(X_s)) dM_s^{u, c} \right).$$

For any $k \in \mathbb{N}$, we define the norm $\|\cdot\|_k$ on the set of measurable functions f from \mathbb{R} into \mathbb{R} by

$$\|f\|_k = \left(\int_E f^2(u_k(x)) \mu_{\langle M^{u_k, c} \rangle}(dx) \right)^{1/2}. \quad (3.3)$$

Let \mathcal{I}_k be the set of measurable functions from \mathbb{R} into \mathbb{R} such that $\|f\|_k < \infty$.

On $\mathcal{I} = \bigcap_{k \in \mathbb{N}} \mathcal{I}_k$, we define a distance d by setting:

$$d(f, g) = [f - g],$$

where

$$[f] = \sum_{k=1}^{\infty} 2^{-k} (1 \wedge \|f\|_k). \quad (3.4)$$

Note that \mathcal{I} contains the measurable locally bounded functions and that the set of elementary functions is dense in (\mathcal{I}, d) . Indeed, by a monotone class argument, we can show that if f is bounded, for any $n \in \mathbb{N}$, there exists f_n elementary such that $\sup_{k \leq n} \|f - f_n\|_k \leq 2^{-n}$. Hence,

$$\sum_{n=1}^{\infty} [f - f_n] \leq \sum_{n=1}^{\infty} \left(\sum_{k=1}^n 2^{-k} (1 \wedge \|f - f_n\|_k) + 2^{-n} \right) < 2.$$

Consequently it is sufficient to show that the set of bounded functions is dense in \mathcal{I} . By dominated convergence, $\lim_{n \rightarrow \infty} [f - (-n) \vee f \wedge n] = 0$ for any $f \in \mathcal{I}$.

Let f be an element of \mathcal{I} . The MAF W^k defined by: $W_t^k = \int_0^t f(u_k(X_s)) dM_s^{u_k, c}$, has finite energy since: $e(W^k) = \frac{1}{2} \|f\|_k^2$. Hence,

$$fu * M_s^{u, c} := \begin{cases} fu_k * M_s^{u_k, c} & \text{for } t < \sigma_{E \setminus D_k}, \\ 0 & \text{for } t \geq \sigma, \end{cases}$$

belongs to $\mathcal{M}_{f\text{-loc}}^c$ ($\mathcal{M}_{\text{loc}}^c$ if $u \in \mathcal{F}$) and by Lemma 2.2, $\Gamma(fu * M^{u, c})$ is well defined and is an element of $\mathcal{N}_{c, f\text{-loc}}$ ($\mathcal{N}_{c, \text{loc}}$ if $u \in \mathcal{F}$).

Theorem 3.3. *The application defined by (3.2) on the set of elementary functions can be extended to the set \mathcal{I} . This extension, denoted by $\int f(z) d_z \Gamma^z$, for f in \mathcal{I} , satisfies:*

- (i) $\int f(z) d_z \Gamma_t^z = \Gamma_t(fu * M^{u, c}) \forall t \geq 0$, \mathbb{P}_x -a.e. for q.e. $x \in E$.
- (ii) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence \mathcal{I} . Assume that: $[f_n - f] \rightarrow 0$. Then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $(\int f_{n_k}(z) d_z \Gamma_t^z)_{k \in \mathbb{N}}$ converges uniformly on any compact of $[0, \zeta)$ ($[0, \infty)$ if $u \in \mathcal{F}$) to $\int f(z) d_z \Gamma_t^z$ \mathbb{P}_x -a.e. for q.e. $x \in E$.

Proof. Elementary functions are dense in \mathcal{I} and (i) holds for elementary functions. It is sufficient to prove that if $[f_n - f]$ converge to zero, there exists a subsequence n_k such that for any $p \in \mathbb{N}$, $\Gamma(f_{n_k} u * M^{u, c})$ converges to $\Gamma(fu * M^{u, c})$ uniformly on any compact of $[0, \sigma_{E \setminus D_p}]$. Let n_k be such that $[f_{n_k} - f] < 2^{-4k}$ and $p \in \mathbb{N}$, hence $\|f - f_{n_k}\|_p \leq 2^p 2^{-4k}$ for any $k > p/4$ and it follows from Theorem 2.5 that $\Gamma(f_{n_k} u_p * M^{u_p, c})$ converges uniformly on any compact to $\Gamma(fu_p * M^{u_p, c})$ \mathbb{P}_x -a.e. for q.e. $x \in E$. But thanks to (2.6), $\Gamma(f_{n_k} u_p * M^{u_p, c})$ and $\Gamma(fu_p * M^{u_p, c})$ agrees on $t < \sigma_{E \setminus D_p}$ with $\Gamma(f_{n_k} u * M^{u, c})$ and $\Gamma(fu * M^{u, c})$, respectively, \mathbb{P}_x -a.e. for q.e. $x \in E$. \square

We finish this section with a characterization of the set \mathcal{I} when u belongs to \mathcal{F} . Let $\mathcal{E}^{(c)}$ be the local part in the Beurling–Deny decomposition for \mathcal{E} (see Theorem 3.2.1 of [12]). $\mathcal{E}^{(c)}$ has the local property, hence with the same argument used to proof Theorems 5.2.1 and 5.2.3 of [1], there exists a function U in $L^1(\mathbb{R}, dx)$ such that for any function F in \mathcal{C}^1 with bounded derivatives f :

$$\mathcal{E}^{(c)}(F(u), F(u)) = \frac{1}{2} \int_{\mathbb{R}} f^2(x) U(x) dx.$$

Then thanks to (2.3) and Lemma 3.2.3 of [12],

$$\int_E f^2(u(x)) \mu_{\langle M^{u,c} \rangle}(dx) = \int_{\mathbb{R}} f^2(x) U(x) dx,$$

hence it follows by a monotone class argument that for any measurable positive function f we have:

$$\int_E f(u(x)) \mu_{\langle M^{u,c} \rangle}(dx) = \int_{\mathbb{R}} f(x) U(x) dx. \quad (3.5)$$

Lemma 3.4. *For u element of \mathcal{F} , the set \mathcal{I} coincides with the set $L^1_{\text{loc}}(\mathbb{R}, U(x) dx)$, where the function U is defined by (3.5).*

Proof. For k integer, the function u_k is defined be $(-k) \vee u \wedge k$. Associate U_k to u_k as U is associated to u . We have then: $\|f\|_k^2 = \int_{\mathbb{R}} f^2(x) U_k(x) dx$ for any measurable function f . In order to proof Lemma 3.4, it is sufficient to prove that: $U_k(x) = 1_{[-k,k]} U(x)$ for a.e. x in \mathbb{R} .

Let f be a continuous function with support in $[-k, k]$ and set $F(x) := \int_0^x f(z) dz$. We have hence: $F(u(x)) = F(u_k(x))$ for any x in E and therefore $f(u_k) * M^{u_k,c} = f(u) * M^{u,c}$, indeed thanks to (2.3) both martingales coincides with $M^{F u_k, c} (= M^{F u, c})$.

We have therefore: $\int_E f^2(u_k(x)) \mu_{\langle M^{u_k,c} \rangle}(dx) = \int_E f^2(u(x)) \mu_{\langle M^{u,c} \rangle}(dx)$. This shows that

$$\int_{\mathbb{R}} f^2(x) U_k(x) dx = \int_{\mathbb{R}} f^2(x) U(x) dx$$

for any function f continuous with compact support in $[-k, k]$, hence $U_k(x) = U(x)$ for a.e. x in $[-k, k]$.

Now if g is a continuous positive function with support in $\mathbb{R} \setminus [-k, k]$ then:

$$\int_{\mathbb{R}} g(x) U_k(x) dx = \int_E g(u_k(x)) \mu_{\langle M^{u_k,c} \rangle}(dx) = 0$$

therefore $U_k(x) = 0$ for a.e. x in $\mathbb{R} \setminus [-k, k]$. This finishes the proof. \square

4. Itô formula

In this section, we first prove Lemma 1.1 and then Theorem 1.2.

Proof of Lemma 1.1. Let u be an element of \mathcal{F}_{loc} , thanks to the proof of Lemma 3.1 of [3], there exists a nest of finely open Borel sets $\{\mathcal{G}_k\}_{k \in \mathbb{N}}$ and a sequence $\{u_k\}_{k \in \mathbb{N}}$ in \mathcal{F} such that $u(x) = u_k(x)$ for q.e. $x \in \mathcal{G}_k$ and $\|u_k\|_\infty < k$. Let $\phi \in L^1(E; m)$ such that $0 < \phi \leq 1$ and for any k let

$$h_k(x) := \mathbb{E}_x \left(\int_0^{\sigma_{E \setminus \mathcal{G}_k}} e^{-t} \phi(X_t) dt \right),$$

$G_k := \{x \in E : h_k(x) > k^{-1}\}$ and $g_k(x) := 1 \wedge (kh_k(x))$. For any k , $G_k \subset \mathcal{G}_k$, thus $u(x) = u_k(x)$ for q.e. $x \in G_k$. Moreover, by the proof of Lemma 3.8 of [13], $\{G_k\}_{k \in \mathbb{N}}$ is a nest and we have: $0 \leq g_k \leq 1$, $g_k(x) = 1$ q.e. on G_k , $g_k(x) = 0$ on $E \setminus \mathcal{G}_k$. Since h_k is quasi-continuous, we can suppose that each G_k is finely open (Theorem 4.6.1 of [12]). For any $k \in \mathbb{N}$, we have:

$$\begin{aligned} & \int_{G_k} \int_{\{|u(x)-u(y)| < 1\}} |u(x) - u(y)|^2 N(x, dy) \nu_H(dx) \\ &= \int_{G_k} |g_k(x)|^2 \int_{\{|u(x)-u(y)| < 1\}} |u(x) - u(y)|^2 N(x, dy) \nu_H(dx) \\ &\leq 2 \int_{G_k} \int_{\{|u(x)-u(y)| < 1\}} |g_k(x) - g_k(y)|^2 |u(x) - u(y)|^2 N(x, dy) \nu_H(dx) \\ &\quad + 2 \int_{\mathcal{G}_k \times \mathcal{G}_k \cap \{|u(x)-u(y)| < 1\}} |g_k(y)|^2 |u(x) - u(y)|^2 N(x, dy) \nu_H(dx) \\ &\leq 2 \int_{E \times E} |g_k(x) - g_k(y)|^2 N(x, dy) \nu_H(dx) \\ &\quad + 2 \int_{E \times E} |u_k(x) - u_k(y)|^2 N(x, dy) \nu_H(dx) \\ &\leq 4\mathcal{E}(g_k, g_k) + 4\mathcal{E}(u_k, u_k) < \infty. \end{aligned}$$

Therefore, if for any $\varepsilon > 0$, we set:

$$S_\varepsilon = \sum_{k=1}^{\infty} 2^{-k} \left(1 \wedge \int_{G_k} \int_{\{|u(x)-u(y)| < \varepsilon\}} |u(x) - u(y)|^2 N(x, dy) \nu_H(dx) \right).$$

We have then $\lim_{\varepsilon \rightarrow 0} S_\varepsilon = 0$. We choose a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $S_{\varepsilon_n} < 2^{-4n}$.

Let F be a locally absolutely continuous function with a locally bounded Radon–Nikodym derivative f . For k in \mathbb{N} , define (F_k) by

$$F_k(x) = F(x)1_{[-k-1, k+1]}(x) + F(k+1)1_{[k+1, \infty)}(x) + F(-k-1)1_{(-\infty, -k-1]}(x).$$

Note that F_k has a bounded Radon–Nikodym derivative: $f_k = f1_{[-k-1, k+1]}$.

For a function $\beta: E^2 \rightarrow \mathbb{R}$, define:

$$A_t(\beta, n) := \int_0^t \int_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \beta(y, X_s) N(X_s, dy) dH_s \quad \text{and}$$

$$M^d(\beta, n) = \sum_{s \leq t} \beta(X_s, X_{s-}) 1_{\{\varepsilon_n < |u(X_{s-}) - u(X_s)| < 1\}} 1_{\{s < \xi\}} - A_t(\beta, n).$$

Denote by $M^d(F, u, n)$ (resp., $M^d(F, u, n, k)$) the process $M^d(\beta, n)$ for $\beta(y, x) = F(u(y)) - F(u(x))$ (resp., $\beta(y, x) = (F(u(y)) - F(u(x))) 1_{G_k}(x)$). Similarly, define $A^d(F, u, n)$ and $A(F, u, n, k)$.

We just have to prove that \mathbb{P}_x -a.e. for q.e. $x \in E$, the limits $\lim_{n \rightarrow \infty} M^d(F, u, n)$ and $\lim_{n \rightarrow \infty} A(F, u, n)$ exist uniformly on any compact of $[0, \sigma_{E \setminus G_k}[$. We have: $M_t^d(F, u, n) = M_t^d(F_k, u, n, k)$ and $A_t(F, u, n) = A_t(F_k, u, n, k)$ on $[0, \sigma_{E \setminus G_k}[$. For every k , the process $M^d(F_k, u, n, k)$ belongs to $\dot{\mathcal{M}}$ and for $4n > k$, we have

$$e(M^d(F_k, u, n+1, k) - M^d(F_k, u, n, k)) \leq c_k 2^k 2^{-4n},$$

where $c_k = \|f_k\|_\infty$. Indeed, from the definition of ε_n :

$$\begin{aligned} & e(M^d(F_k, u, n+1, k) - M^d(F_k, u, n, k)) \\ &= \frac{1}{2} \int_{G_k \times E} (F_k(u(x)) - F_k(u(y)))^2 1_{\{\varepsilon_{n+1} \leq |u(x) - u(y)| < \varepsilon_n\}} N(x, dy) \nu_H(dx) \\ &\leq c_k \int_{G_k \times E} |u(x) - u(y)|^2 1_{\{|u(x) - u(y)| < \varepsilon_n\}} N(x, dy) \nu_H(dx) \\ &\leq c_k 2^k 2^{-4n} \end{aligned}$$

thus, the convergence of $M^d(F, u, n)$ follows from Theorem 2.5. Still thanks to Theorem 2.5, the convergence of $A(F, u, n)$ can be seen as a consequence of:

$$\Gamma(M_t^d(F_k, u, n, k)) = A_t(F_k, u, n, k), \quad \mathbb{P}_x\text{-a.e. for q.e. } x \in E. \quad (4.1)$$

To prove (4.1), we note that $(A_t(F_k, u, n, k))_{t \geq 0}$ is of bounded variation, so $A_t(F_k, u, n, k) \circ r_t = A_t(F_k, u, n, k)$ \mathbb{P}_m -a.e. on $t < \zeta$ (Theorem 2.1 of [11]). Hence, making use of the operator Λ defined in [3], instead of Γ , we first obtain:

$$\Lambda(M_t^d(F_k, u, n, k)) = A_t(F_k, u, n, k), \quad \mathbb{P}_m\text{-a.e. for q.e. } x \in E \text{ on } \llbracket 0, \zeta \rrbracket.$$

Finally by Theorem 3.6 in [3] and Lemma 2.1, (4.1) holds, \mathbb{P}_x -a.e. for q.e. $x \in E$ on $\llbracket 0, \zeta \rrbracket$, and therefore on $\llbracket 0, \infty \rrbracket$ thanks to the continuity of $\Gamma(M_t^d(F_k, u, n, k))$ and $A_t(F_k, u, n, k)$.

It is clear that $M^d(F, u) \in \dot{\mathcal{M}}_{f\text{-loc}}$ and $A(F, u) \in \mathcal{N}_{c, f\text{-loc}}$. Moreover, for u element of \mathcal{F} , we can take $G_n = \{x: |u(x)| < n\}$ for any n . In this case, from the strict continuity of u we have, $\mathbb{P}_x(\lim_{n \rightarrow \infty} \sigma_{E \setminus G_n} = \infty) = 1$ for q.e. $x \in E$, thus the convergence of $M^d(F, u, n)$

and $A(F, u, n)$ are uniformly on any compact of $[0, \infty)$. Thus, we obtain: $M^d(F, u) \in \mathring{\mathcal{M}}_{\text{loc}}$ and $A(F, u) \in \mathcal{N}_{c, \text{loc}}$. \square

Remark 4.1. (i) If $u \in \mathcal{F}$ and f is bounded, then $M^d(F, u) \in \mathring{\mathcal{M}}$ and $\Gamma(M^d(F, u)) = A(F, u)$. (ii) With the notation of the proof of Lemma 1.1, it holds that if $u_k = u$ q.e. on G_k :

$$M_t^d(F, u) + A_t(F, u) = M_t^d(F_k, u_k) + A_t(F_k, u_k) \quad \text{for } t \in [0, \sigma_{E \setminus G_k}[, \mathbb{P}_x\text{-a.e. for q.e. } x \in E.$$

Proof of Theorem 1.2. We use the notation of the proof of Lemma 1.1. Thus, if $u \in \mathcal{F}$, we take $G_n := \{x: |u(x)| < n\}$, $n \in \mathbb{N}$. Let F be a locally absolutely continuous function F with a locally bounded Radon–Nikodym derivative f .

Let I_t be the difference of the left-hand side and the right-hand side of (1.3). For any k , we define I_t^k as I_t with u_k and f_k replacing u and f , respectively. Hence, $I_t = I_t^k$ for $t < \sigma_{E \setminus G_k}$, \mathbb{P}_x -a.e. for q.e. $x \in E$. Since $\sigma_{E \setminus G_n} \wedge \zeta$ converges to ζ if $u \in \mathcal{F}_{\text{loc}}$ and $\sigma_{E \setminus G_n}$ converges to ∞ if $u \in \mathcal{F}$, it is sufficient to prove (1.3) on $[0, \sigma_{E \setminus G_k}[$ for any $k \in \mathbb{N}$. Consequently, we can assume (and we do) that u is an element of \mathcal{F}_b and f is bounded.

If f is continuous, thanks to (2.3), $F(u) \in \mathcal{F}$ and $M^{F, u, c} = fu * M^{u, c}$ and we have the Fukushima decomposition:

$$F(u(X_t)) = F(u(X_0)) + fu * M_t^{u, c} + \Gamma(fu * M^{u, c})_t + M_t^{u, d} + \Gamma(M^{u, d})_t.$$

We obtain (1.3) from Lemma 3.3(i) and Remark 4.1(i).

If f is not necessarily continuous, let g be in $L^1(\mathbb{R})$ be a strictly positive function on \mathbb{R} such that g and $1/g$ are locally bounded. Define the norms $\|\cdot\|$ and $\|\cdot\|_*$ on the Borel measurable functions as follows:

$$\begin{aligned} \|h\|_* &= \left(\int_E h^2(u(x)) \mu_{\langle M^{u, c} \rangle}(\mathrm{d}x) \right)^{1/2}, \\ \|h\| &= \|h\|_* + \int |h(x)|g(x) \mathrm{d}x \\ &\quad + \left(\int_{E \times E - \delta} |u(x) - u(y)| \int_{u(x) \wedge u(y)}^{u(x) \vee u(y)} h(z)^2 \mathrm{d}z N(x, \mathrm{d}y) \nu_H(\mathrm{d}x) \right)^{1/2}. \end{aligned}$$

Since u is in \mathcal{F} , we have $\|f\| < \infty$. By a monotone class argument, one shows that there exists a sequence of bounded continuous functions $(f_n)_{n \in \mathbb{N}}$ with compact support such that $\|f_n - f\|$ converges to 0 as n tends to infinity. We set $F_n(x) = \int_0^x f_n(z) \mathrm{d}z$.

In order to show (1.3), we will show that there exists a subsequence n_k such that the terms in the expansion (1.3) for F_{n_k} converge as $k \rightarrow \infty$ to the corresponding expression with f replacing f_{n_k} . The convergence of $F_n(u(X_t)) - F_n(u(X_0)) - V_t(F_n, u)$ to $F(u(X_t)) - F(u(X_0)) - V_t(F, u)$ is a consequence of the pointwise convergence of F_n

to F , indeed, for any $x \in \mathbb{R}$,

$$|F_n(x) - F(x)| \leq \int_{-x^-}^{x^+} |f_n(z) - f(z)| dz \leq \sup_{|\lambda| \leq |x|} \frac{1}{g(\lambda)} \int_{-\infty}^{\infty} |f_n(z) - f(z)| g(z) dz \rightarrow 0.$$

The existence of a subsequence $\{n_k\}$ such that $\int_0^t f_{n_k}(u(X_s)) dM_s^{u,c}$ and $\int_{\mathbb{R}} f_{n_k}(z) dz \Gamma_t^z(u)$ converge to $\int_0^t f(u(X_s)) dM_s^{u,c}$ and $\int_{\mathbb{R}} f(z) dz \Gamma_t^z(u)$, respectively, is a consequence of the fact that $e(fu * M^{u,c} - f_n u * M^{u,c}) = \frac{1}{2} \|f - f_n\|_*^2 \rightarrow 0$ as $n \rightarrow \infty$, and Theorem 2.5. Thanks to Theorem 2.5 and Remark 4.1(i), it is then sufficient to show that $e(M(F_n, u) - M(F, u))$ converges to zero as $n \rightarrow \infty$. But

$$\begin{aligned} e(M - M^n) &\leq \frac{1}{2} \int_{E \times E - \delta} (F(u(x)) - F_n(u(x)) - F(u(y)) \\ &\quad + F_n(u(y)))^2 N(x, dy) \nu_H(dx) \\ &\leq \frac{1}{2} \|f - f_n\|_*^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad \square$$

As an example, for $F(z) = z$ and u in \mathcal{F}_{loc} , one obtains a Fukushima decomposition for the process $u(X)$. This case can be seen as a refinement of Lemma 2.2 in [4].

5. Local time

We fix an element u of \mathcal{F}_{loc} . The associated process ${}^cN^u$ has been defined in (2.1) by ${}^cN^u = \Gamma(M^{u,c})$. By Remark 2.3, ${}^cN^u$ is a CAF locally of zero energy or merely a CAF of zero energy when u belongs to F . We suppose that u satisfies the additional assumption that ${}^cN^u$ is of bounded variation on $[0, \zeta)$, that is, there exists two PCAF's $A^{(1)}$ and $A^{(2)}$ such that \mathbb{P}_x -a.e. for q.e. $x \in E$:

$${}^cN_t^u = A_t^{(1)} - A_t^{(2)} \quad \forall t \in [0, \zeta). \quad (5.1)$$

We remind that a measure ν on E is a smooth signed measure on E if there exists a nest $\{F_k\}$ such that for each k , $1_{F_k} \cdot \nu$ is a finite signed Borel measure charging no set of zero capacity and further ν charges no Borel subset of $E \setminus \bigcup_{k=1}^{\infty} F_k$. Such nest is said to be associated to ν . For a closed set $F \subset E$, we set:

$$\mathcal{F}_{b,F} = \{u \in \mathcal{F}_b : u = 0 \text{ q.e. on } E \setminus F\}.$$

We also need the following definition:

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v)_m.$$

Lemma 5.1. *The process ${}^cN^u$ is of bounded variation if and only if there exists a smooth signed measure ν on E with associated nest $\{F_k\}$ such that*

$$\mathcal{E}^{(c)}(u, v) = \langle v, \nu \rangle, \quad \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b, F_k}.$$

Proof. From Theorem 5.2.4 of [12], ${}^cN^u$ is the only AF of zero energy such that for any $h \in \mathcal{F}$,

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{h, m} [{}^cN_t^u] = -e(M^{u, c}, M^{h, c}) = -\mathcal{E}^{(c)}(u, h).$$

On the other hand, since: $|\mathcal{E}^{(c)}(u, h)| \leq (\mathcal{E}^{(c)}(u, u))^{1/2} (\mathcal{E}_1(h, h))^{1/2}$, there exists a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}^{(c)}(u, h) = \mathcal{E}_1(w, h) \quad \text{for any } h \in \mathcal{F}.$$

Hence, $\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{h, m} [N_t^w - \int_0^t w(X_s) ds] = -\mathcal{E}^{(c)}(u, h)$ for all $h \in \mathcal{F}$. This implies that the AF $N^w - \int_0^\cdot w(X_s) ds$ is equivalent to ${}^cN^u$. Consequently, ${}^cN^u$ is of bounded variation if and only if N^w is of bounded variation. But thanks to Theorem 5.4.2 of [12], this last condition is equivalent to the existence of a smooth signed measure ν with an associated nest $\{F_k\}$ such that

$$\mathcal{E}_1(w, v) = \langle v, \nu \rangle \quad \forall v \in \bigcup_{k=1}^{\infty} \mathcal{F}_{b, F_k}. \quad \square$$

5.1. Definition of local time

Definition 5.2. *The local time at a of $u(X)$, denoted by $L_t^a = L_t^a(u)$ is the following CAF on $\llbracket 0, \zeta \rrbracket$:*

$$\frac{1}{2} L_t^a = -\Gamma(Z^a)_t + \int_0^t 1_{\{u(X_{s-}) \leq a\}} d^c N_s^u \quad \text{for } t \in [0, \zeta].$$

The name “local time” is justified by Proposition 5.3 and Corollary 5.4 below.

Proposition 5.3. *There exists a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$ -measurable version of the local time process $\{\tilde{L}_t^a; a \in \mathbb{R}, t \geq 0\}$ such that \mathbb{P}_m -a.e. we have the occupation time density formula:*

$$\int_{\mathbb{R}} f(x) \tilde{L}_t^x dx = \int_0^t f(u(X_s)) d\langle M^{u, c} \rangle_s \quad \text{for any } f \text{ Borel bounded and } t < \zeta.$$

Proof. We start with the case when u is a bounded element of \mathcal{F} . From (2.4) we have: \mathbb{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket$: $L_t^a = Z_t^a + Z_t^a \circ r_t + 2 \int_0^t 1_{\{u(X_{s-}) \leq a\}} dN_s^{u, c}$. Moreover, thanks to Theorem 63, Chapter IV of [15], there exists a function $\tilde{Z}(a, t, \omega)$ in $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$,

such that for each $a \in \mathbb{R}$, $\tilde{Z}(a, t, w)$ is a continuous (\mathcal{F}_t^m) -adapted version of the stochastic integral Z^a , and thanks to Lemma 2.10 and Theorem 2.18 of [3], $\tilde{Z}(a, t, w) \circ r_t(\omega) \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$ is a continuous (\mathcal{F}_t^m) -adapted version of $Z_t^a \circ r_t$ for each $a \in \mathbb{R}$. Besides, we can take $\int_0^t 1_{\{u(X_{s-}) \leq a\}} dN_s^{u,c}$ jointly continuous in t and right continuous in a , \mathbb{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket \times \mathbb{R}$. Thus, we have constructed a version $\{\tilde{L}_t^a, a \in \mathbb{R}, t \in [0, \zeta[\}$ of $\{L_t^a, a \in \mathbb{R}, t \in [0, \zeta[\}$ which is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_\infty^m$ -measurable.

Let f be a continuous positive element of $L^1(\mathbb{R})$. Using the proof presented in [15] of Fubini's theorem for stochastic integrals (Theorem 64, Chapter IV of [15]), we know that $\int_{\mathbb{R}} \tilde{Z}(z, t, \omega) f(z) dz$ is a well-defined Lebesgue integral since \mathbb{P}_m -a.e.:

$$\int_{\mathbb{R}} |\tilde{Z}(z, t, \omega)| f(z) dz < \infty \quad \text{for all } t.$$

Moreover, still thanks to this theorem, $\int_{\mathbb{R}} \tilde{Z}(z, t, \omega) f(z) dz$ is a continuous \mathbb{P}_m -version of $\int_0^t F(u(X_s)) dM_s^{u,c}$, where $F(z) = \int_z^\infty f(\lambda) d\lambda$. Consequently, for $t > 0$, \mathbb{P}_m -a.e. on $\{t < \zeta\}$, $\int_{\mathbb{R}} |\tilde{Z}(z, t, r_t(\omega))| f(z) dz < \infty$ and $\int_{\mathbb{R}} \tilde{Z}(z, t, r_t(\omega)) f(z) dz$ is a continuous \mathbb{P}_m -version of $\int_0^t F(u(X_s)) dM_s^{u,c} \circ r_t$.

Since $(\int_0^t 1_{\{u(X_{s-}) \leq a\}} dN_s^{u,c})_{a \in \mathbb{R}}$ is of bounded variation on $\{t < \zeta\}$, we obtain \mathbb{P}_m -a.e. on $\{t < \zeta\}$: $\int_{\mathbb{R}} f(z) |\tilde{L}_t^z| dz < \infty$ and

$$\int_{\mathbb{R}} f(z) \tilde{L}_t^z dz = \int_0^t F(u(X_s)) dM_s^{u,c} + \int_0^t F(u(X_s)) dM_s^{u,c} \circ r_t + 2 \int_0^t F(u(X_s)) dN_s^{u,c}$$

which leads to

$$\int_{\mathbb{R}} f(z) \tilde{L}_t^z dz = -2\Gamma(Fu * M^{u,c})_t + 2 \int_0^t F(u(X_s)) dN_s^{u,c}. \quad (5.2)$$

Now thanks to (2.3), Fu belongs to \mathcal{F}_{loc} and $M_t^{Fu,c} = -\int_0^t f(u(X_s)) dM_s^{u,c}$. Thus,

$$\langle M^{Fu,c}, M^{u,c} \rangle_t = - \int_0^t f(u(X_s)) d\langle M^{u,c} \rangle_s.$$

Thanks to Lemma 2.4 we have \mathbb{P}_m -a.e. on $\{t < \zeta\}$:

$$\int_0^t F(u(X_s)) d^c N_s^u = \int_0^t F(u(X_s)) d\Gamma(M^{u,c})_s.$$

On the other hand, the definition of the integral with respect to $\Gamma(M^{u,c})$ (Chen *et al.* [3]) gives:

$$\int_0^t F(u(X_s)) d\Gamma(M^{u,c})_s = \Gamma(Fu * M^{u,c})_t + \frac{1}{2} \int_0^t f(u(X_s)) d\langle M^{u,c} \rangle_s$$

which together with (5.2) lead to

$$\int_{\mathbb{R}} f(z) \tilde{L}_t^z dz = \int_0^t f(u(X_s)) d\langle M^{u,c} \rangle_s, \quad \mathbb{P}_m\text{-a.e. on } \{t < \zeta\}. \quad (5.3)$$

Actually, the set of null \mathbb{P}_m -measure on which (5.3) could fail can be chosen independently of f . Indeed, the set of continuous functions with compact support, is a separable topological space for the metric of uniform convergence.

We show now that the set of null \mathbb{P}_m -measure on which (5.3) could fail does not depend on t either. We have thanks to (5.3)

$$\mathbb{P}_m\text{-a.e. on } \{t < \zeta\}, \quad \tilde{L}_t^z \geq 0 \text{ for } dz\text{-a.e. } z \quad (5.4)$$

hence by a monotone class argument, (5.3) holds \mathbb{P}_m -a.e. on $\{t < \zeta\}$ for any f Borel bounded. It remains to show that (5.3) holds \mathbb{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket$. To do so, it is sufficient to show that the left-hand side of (5.3) is continuous in t .

It follows from Theorem 2.18 in [3] that for any z , $\tilde{Z}(z, t, r_t(\omega))$ is continuous and has the additivity property \mathbb{P}_m -a.e. for on $\llbracket 0, \zeta \rrbracket$. Hence, thanks to (5.4) for dz -a.e. z , \tilde{L}_t^z is increasing. One shows then by monotone convergence that for any positive Borel function f , $t \rightarrow \int_{\mathbb{R}} f(z) \tilde{L}_t^z dz$ is continuous \mathbb{P}_m -a.e. on $\llbracket 0, \zeta \rrbracket$.

For a function u in \mathcal{F}_{loc} , take an nest of closed sets $\{D_k\}$ and a sequence $(u_k)_{k \in \mathbb{N}}$ of bounded elements of \mathcal{F} such that $u = u_k$ for q.e. $x \in E$. For any $k \in \mathbb{N}$, let $\tilde{L}_t^z(u_k)$ be the version $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_{\infty}^m$ -measurable of local time obtained above. Then $\tilde{L}_t^z := \tilde{L}_t^z(u_k)$ on $t < \tau_{D_k}$ is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_{\infty}^m$ -measurable version of L_t^z and satisfies the occupation time density formula on $[0, \tau_{D_k}[$, for any $k \in \mathbb{N}$, so it satisfies it on $[0, \zeta[$. \square

Corollary 5.4. *For any real a , L^a is a PCAF and \mathbb{P}_x -a.e. for q.e. $x \in E$, the measure in t , $d_t L_t^a$ is carried by the set $\{s: u(X_{s-}) = u(X_s) = a\}$.*

Proof. We use u_k and $\{D_k\}$ defined as in the end of the proof of Proposition 5.3. Since we need to show the assertion of Corollary 5.3 only on $[0, \tau_{D_k}[$, we can assume that u is a bounded element of \mathcal{F} . It follows from the occupation time density formula and the $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}_{\infty}^m$ -measurability of \tilde{L} , that there exists a subset R of \mathbb{R} of Lebesgue's measure zero, such that for any a outside of R : \mathbb{P}_m -a.e. $\tilde{L}_t^a \geq 0$ on $\llbracket 0, \zeta \rrbracket$. Consequently, L^a has the same property. This property holds for any $a \in \mathbb{R}$. Indeed for any real a , take a sequence $(a_n)_{n \in \mathbb{N}} \subset \mathbb{R} \setminus R$ such that $a_n \downarrow a$. We have: $e(Z^{a_n} - Z^a) = \int 1_{\{a < u(x) \leq a_n\}} \mu_{\langle M^{u,c} \rangle}(dx)$, which converges to 0 as n tends to ∞ by dominated convergence. Thus, thanks to Theorem 2.5 (taking a subsequence if necessary) $\Gamma(Z^{a_n})$ converges to $\Gamma(Z^a)$ uniformly on any finite interval of t , \mathbb{P}_m -a.e. On the other hand, for \mathbb{P}_m -a.e. $\omega \in \Omega$, $\int_0^t 1_{\{u(X_s) \leq a_n\}} dN_s^{u,c}(\omega)$ converges to $\int_0^t 1_{\{u(X_s) \leq a\}} dN_s^{u,c}(\omega)$ for any $t < \zeta(\omega)$. Consequently, we obtain for \mathbb{P}_m -a.e. $\omega \in \Omega$, $L_t^a(\omega) \geq 0$ for any $t < \zeta(\omega)$.

It follows from Lemma 2.1 that for any real a , L^a is a PCAF on $\llbracket 0, \zeta \rrbracket$. By Remark 2.2 in [3], it can be extended to a PCAF.

Now defining $f(x) = (x - a)^4$ and $h(x) = (x - a)^4 1_{\{x \leq a\}}$, it follows from (2.3) that fu and hu belong to \mathcal{F}_{loc} . Moreover, we have:

$$M_t^{fu,c} = 4 \int_0^t (u(X_s) - a)^3 dM_s^{u,c} \quad \text{and} \quad M_t^{hu,c} = 4 \int_0^t (u(X_s) - a)^3 1_{\{u(X_s) \leq a\}} dM_s^{u,c}$$

thus, $\langle M^{fu,c}, Z^a \rangle = \langle M^{hu,c}, M^{u,c} \rangle$, and from the definition of the stochastic integral (2.5) we have that \mathbb{P}_m -a.e. on $\{t < \zeta\}$

$$\int_0^t (u(X_s) - a)^4 d\Gamma(Z^a)_s = \int_0^t (u(X_s) - a)^4 1_{\{u(X_s) \leq a\}} d\Gamma(M^{u,c})_s.$$

By Lemmas 2.1 and 2.4, we finally obtain: $\int_0^t (u(X_s) - a)^4 dL_s^a = 0$ \mathbb{P}_x -a.e. for q.e. $x \in E$. \square

5.2. Integration with respect to local time

We fix u an element of \mathcal{F} satisfying (5.1) and set: $l_t^a = \int_0^t 1_{\{u(X_{s-}) \leq a\}} dN_s^{u,c}$. Hence, the local time at a of $u(X)$ satisfies:

$$L^a = -2\Gamma^a + 2l^a.$$

For any $\omega \in \Omega$ and $t < \zeta(\omega)$, the function $z \rightarrow l_t^z(\omega)$ is of bounded variation. The application defined for the elementary functions by

$$f \rightarrow \sum_{i=0}^{n-1} f_i(l_t^{z_{i+1}} - l_t^{z_i}), \quad t < \zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from \mathbb{R} into \mathbb{R} as a Lebesgue–Stieljes integral and we have:

$$\int_{\mathbb{R}} f(z) dz l_t^z = \int_0^t f(u(X_s)) dN_s^{u,c}, \quad t < \zeta.$$

Using the stochastic integral with respect to Γ , the application defined for the elementary functions by

$$f \rightarrow \sum_{i=0}^{n-1} f_i(L_t^{z_{i+1}} - L_t^{z_i}), \quad t < \zeta$$

can hence be extended to the set of locally bounded Borel measurable functions f from \mathbb{R} into \mathbb{R} and we have:

$$-\frac{1}{2} \int_{\mathbb{R}} f(z) dz L_t^z = \int_{\mathbb{R}} f(z) dz \Gamma_t^z - \int_0^t f(u(X_s)) dN_s^{u,c}, \quad t < \zeta.$$

6. Multidimensional case

In this section, we need the following notation. For $d \in \mathbb{N}$, $x = (x^1, \dots, x^d), y = (y^1, \dots, y^d) \in \mathbb{R}^d$, we set $x \leq y$ (resp., $x < y$) if and only if $x^i \leq y^i$ (resp., $x^i < y^i$) for each $i = 1, \dots, d$ and $]x, y] = \{z \in \mathbb{R}^d: x < z \leq y\}$. The vector \hat{x} is obtained from x by elimination of its coordinate x^d , that is, $\hat{x} = (x^1, \dots, x^{d-1})$, $]\hat{x}, \hat{y}] = \{z \in \mathbb{R}^{d-1}: \hat{x} < z \leq \hat{y}\}$.

Let φ be a measurable function from \mathbb{R}^d into \mathbb{R} . We define integration of simple functions with respect to φ as follows. For f a simple function, that is, there exists $x, y \in \mathbb{R}^d$ such that $f(z) = 1_{]x, y]}(z)$ for all $z \in \mathbb{R}^d$:

$$\begin{aligned} \text{if } d = 1: \quad & \int_{\mathbb{R}} f(z) d\varphi(z) = \varphi(y) - \varphi(x), \\ \text{if } d > 1: \quad & \int_{\mathbb{R}^d} f(z) d\varphi(z) = \int_{\mathbb{R}^{d-1}} 1_{]\hat{x}, \hat{y}]}(z) d\varphi(z, y^d) - \int_{\mathbb{R}^{d-1}} 1_{]\hat{x}, \hat{y}]}(z) d\varphi(z, x^d). \end{aligned}$$

As an example, if there exist functions h_i , $1 \leq i \leq d$ such that $\varphi(z) = \prod_{i=1}^d h_i(z^i)$, then $\int_{\mathbb{R}^d} f(z) d\varphi(z) = \prod_{i=1}^d (h_i(y^i) - h_i(x^i))$.

We extend this integration to the elementary functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ (i.e., $f(z) = \sum_{i=1}^n a_i f_i(z)$ where f_i , $1 \leq i \leq n$, are simple functions and a_i , $1 \leq i \leq n$, are real numbers) by setting

$$\int_{\mathbb{R}^d} f(z) d\varphi(z) = \sum_{i=1}^n a_i \int_{\mathbb{R}^d} f_i(z) d\varphi(z).$$

An elementary function has many representations as linear combination of simple functions, but as in the Riemann integration theory, the integral does not depend on the choice of its representation.

Let u be in $\mathcal{F}_{\text{loc}}^d$ where $\mathcal{F}_{\text{loc}}^d = \{(u^1, u^2, \dots, u^d): u^i \in \mathcal{F}_{\text{loc}}, 1 \leq i \leq d\}$. Let $\{D_k\}_{k \in \mathbb{N}}$ be a nest of closed set, $\sigma := \lim_{k \rightarrow \infty} \sigma_{E \setminus D_k}$ and $(u_k)_{k \in \mathbb{N}}$ a sequence of bounded elements of \mathcal{F}^d such that $u = u_k$ q.e. on D_k .

For any a in \mathbb{R}^d and i in $\{1, 2, \dots, d\}$, we define $Z^a(u^i)$ and $\Gamma^a(u^i)$, respectively, in $\mathcal{M}_{f\text{-loc}}^c$ and $\mathcal{N}_{c, f\text{-loc}}$ by

$$\begin{aligned} Z_t^a(u^i) &= \begin{cases} \int_0^t 1_{\{u_k(X_{s-}) \leq a\}} dM_s^{u_k^i, c} & \text{for } t \leq \sigma_{E \setminus D_k}, \\ 0 & \text{for } t \geq \sigma, \end{cases} \\ \Gamma^a(u^i) &= \Gamma(Z^a(u^i)). \end{aligned}$$

Thanks to the linearity property of Γ , we have for any elementary function f :

$$\int_{\mathbb{R}^d} f(z) dz \Gamma_t^z(u^i) = \Gamma_t \left(\int_0^t f(u(X_s)) dM_s^{u^i, c} \right).$$

We extend (3.3) of Section 3 from $d = 1$ to $d \geq 1$, by defining for $k \in \mathbb{N}$, the norm $\|\cdot\|_k$ on the set of measurable functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$

$$\|f\|_k := \sum_{i=1}^d \left(\int_E f^2(u_k(x)) \mu_{\langle M^{u_k^i, c} \rangle}(dx) \right)^{1/2}$$

and we define the set \mathcal{I} with the metric $[\cdot, \cdot]$ as in (3.4) of Section 3. The set of elementary functions is dense in \mathcal{I} . We have the following version of Lemma 3.3.

Lemma 6.1. *The applications $f \rightarrow \int_{\mathbb{R}^d} f(z) dz \Gamma_t^z(u^i)$ ($1 \leq i \leq d$) defined on the set of elementary functions, can be extended to the set \mathcal{I} . This extensions, denoted by $\int_{\mathbb{R}^d} dz \Gamma^z(u^i)$, satisfy:*

- (i) $\int_{\mathbb{R}^d} f(z) dz \Gamma_t^z(u^i) = \Gamma(fu * M^{u^i, c})_t \quad \forall t \geq 0, \mathbb{P}_x\text{-a.e. for q.e. } x \in E.$
- (ii) *For $(f_n)_{n \in \mathbb{N}}$ sequence of \mathcal{I} such that $[f_n - f] \rightarrow 0$, there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $\int f_{n_k}(z) dz \Gamma_t^z(u^i)$ converges uniformly on any compact of $[0, \zeta)$ ($[0, \infty)$ if $u \in \mathcal{F}^d$) to $\int f(z) dz \Gamma_t^z(u^i)$ for every $1 \leq i \leq d$ $\mathbb{P}_x\text{-a.e. for q.e. } x \in E.$*

With can prove a multidimensional version of Lemma 1.1 with the same arguments used in its proof. We have the following multidimensional Itô formula.

Proposition 6.2. *Let u be an element of \mathcal{F}^d (resp., $\mathcal{F}_{\text{loc}}^d$) and $F: \mathbb{R}^d \rightarrow \mathbb{R}$ a continuous function admitting locally bounded Radon–Nikodym derivatives $f_i = \partial F / \partial x_i$, $1 \leq i \leq d$, satisfying the following condition for any $1 \leq i \leq d$ and $k \in \mathbb{N}$*

$$\lim_{h \rightarrow 0} \int_E \{f_i(u_k(x) + h) - f_i(u_k(x))\}^2 \mu_{\langle M^{u_k^i, c} \rangle}(dx) = 0. \quad (6.1)$$

Then, $\mathbb{P}_x\text{-a.e. for q.e. } x \in E$, the process $F(u(X_t)), t \in [0, \infty)$ (resp., $[0, \zeta)$) admits the decomposition

$$F(u(X_t)) = F(u(X_0)) + M_t(F, u) + Q_t(F, u) + V_t(F, u), \quad (6.2)$$

where $M(F, u) \in \mathring{\mathcal{M}}_{\text{loc}}$, (resp., $\mathring{\mathcal{M}}_{f\text{-loc}}$) $Q(F, u) \in \mathcal{N}_{c, \text{loc}}$ (resp., $\mathcal{N}_{c, f\text{-loc}}$) and $V(F, u)$ is a bounded variation process given by:

$$\begin{aligned} M_t(F, u) &= M_t^d(F, u) + \sum_{i=1}^d \int_0^t f_i(u(X_s)) dM_s^{u^i, c}, \\ Q_t(F, u) &= \sum_{i=1}^d \int_{\mathbb{R}} f_i(z) dz \Gamma_t^z(u_i) + A_t(F, u), \\ V_t(F, u) &= \sum_{s \leq t} \{F(u(X_s)) - F(u(X_{s-}))\} 1_{\{|u(X_s) - u(X_{s-})| \geq 1\}} 1_{\{s < \xi\}} \\ &\quad - F(u(X_{\xi-})) 1_{\{t \geq \xi\}}. \end{aligned}$$

Proof. As in the proof of Theorem 1.2, we can assume that u is a bounded element of \mathcal{F} and each f_i is bounded. For $\phi: \mathbb{R}^d \rightarrow \mathbb{R}$ an infinitely differentiable function with compact support, the function F_n defined by $F_n(z) := \int_{\mathbb{R}^d} F(z + y/n) \phi(y) dy$ converges pointwise to $F(z)$. Setting: $f_{n,i} = \partial F_n / \partial x_i$ we obtain thanks to (6.1):

$$\lim_{n \rightarrow \infty} \int_E [f_{n,i}(u(x)) - f_i(u(x))]^2 \mu_{\langle M^{u^i, c} \rangle}(dx) = 0.$$

The rest of the proof follows step by step the proof of Theorem 1.2. \square

In the case where $E = \mathbb{R}^d$ and $\mathcal{E}^{(c)}$ is given by

$$\mathcal{E}^{(c)} = \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a_{ij}(x) dx,$$

where for every (i, j) , a_{ij} is a bounded measurable function. The coordinates functions $\pi_i(x) = x_i, 1 \leq i \leq d$, belong to \mathcal{F}_{loc} and $M = (M^{\pi_1, c}, \dots, M^{\pi_d, c})$ is a martingale additive functional with quadratic covariation $\langle M^i, M^j \rangle_s = \int_0^t a_{ij}(X_s) ds$, hence, $\mu_{\langle M^{i, c} \rangle}(dx) = a_{ii}(x) dx$, and the condition (6.1) holds for any locally bounded measurable function.

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